Differential Geometry

B. Math. III

Semstral Examination

Instructions: All questions carry equal marks.

1. Let $\alpha : I \to \mathbb{R}^3$ be a regular space curve parametrised by arc length with nowhere zero curvature. Assume that all the normals of α pass through a fixed point in $\mathbb{R}^3$. Prove that the curve is planar and its image is contained in a circle.

2. Let $\kappa > 0$ and τ be two real numbers. Describe all regular curves in $\mathbb{R}^3$ having curvature κ and torsion τ.

3. Let I be an open interval in $\mathbb{R}$ and $\alpha : I \to \mathbb{R}^3$ be a regular parametrized space curve with nowhere zero curvature. Let
$$S = \{v \in \mathbb{R}^3 \mid v = \alpha(t) + y\alpha'(t) \text{ for some } t \in I,\ 0 \neq y \in \mathbb{R}\}$$
Show that S is a regular surface (called the *tangent surface* of α) in $\mathbb{R}^3$ whose tangent planes at the points on the curve that is the image of the line $t = \text{const}$ in S are all equal.

4. Let $S \subset \mathbb{R}^3$ be an oriented regular surface. Define the Gauss map of S and prove that its differential at each point $p \in S$ is a self-adjoint (with respect to the First Fundamental Form of S) linear operator on the tangent space $T_p(S)$ of S at p.

5. Let $S \subset \mathbb{R}^3$ be an oriented surface in and let $\alpha : I \to S$ be a smooth regular curve. Define the normal and geodesic curvature of α by explaining in full detail all the involved terms.